PART I: BASIC INFORMATION

Lesson Topic: Average rates of change in context.

Discipline or Field: Mathematics.

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Lesson Site: University of Wisconsin at River Falls

Course Name: Calculus I

Course Description:
This is a first semester calculus course, covering limits, derivatives, and an introduction to integrals, but not including exponential functions. It is the first required course for Math majors, and is required by many other majors, mostly in the sciences. Class size is typically about 30 students, and classes are 55 minutes long. The setting is a typical classroom, equipped with desks, blackboards and an overhead projector. This lesson is designed to be used early in the lessons about derivatives: following the limit definitions, but preceding the typical calculus lesson on rates of change in context. This lesson would also be appropriate for use just before introducing derivatives.

Summary:
Our goal is for students to better understand rate of change in context, including the skills of moving flexibly between algebraic and graphical representations and analyzing behaviors given information about the rate of change. In this lesson, students practice these skills in concrete examples using average rate of change, as a preparation for doing similar work with derivatives. These activities are at an appropriate level, with some review, and some critical thinking work, and they prompt valuable discussion among students about rates of change.

PART II: THE LESSON

Learning Goals:
The goal of this lesson is for students to have a better grasp of rate of change in context, with the big picture being to encourage students to move flexibly between algebraic and graphical representations in contextual problems, and to analyze behaviors from information about the rate of change. This lesson addresses these goals by having students make connections between rate of change calculations, and their geometric interpretations as slope of an appropriate graph in concrete examples of average rate of change, as a preparation for doing similar work with derivatives.

Specifically, in a concrete and intuitive context, students will:

- Explain the important differences between change and rate of change
- Identify the geometric interpretation of rate of change as slope of a line
- Identify appropriate units for rate of change in context
- Explain the contextual behavior associated with different rates of change

Lesson Design: Brief summary:

The teacher introduces the rate of change activity sheets, and prompts students to suggest ways that they might begin solving the assignment. (5 minutes)

Students work in groups on the first page of the worksheets: table and calculations for change and rate of change (10 minutes)

Teacher leads discussion, highlighting important concepts from the first page (5 minutes)

Students work in groups on the second page: graphing rate of change, and units of rate of change (5 minutes)

Teacher leads discussion of the second page (5 minutes)

Students work in groups on the third page: connections between algebraic, graphic and contextualized information (5-10 minutes)

Teacher leads discussion of third page (5-10 minutes)

Teacher summarizes the important concepts, and introduces the homework (10 minutes)

A detailed walkthrough including sample teacher questions and typical student errors is provided in the lesson document: Rate_of_change.doc.

Rationale:

We chose the topic of rate of change, because in our experience, the process of moving from a single way of looking at derivatives (slope of a tangent line) to a broader contextual way of looking at derivatives (instantaneous rate of change) is often a difficult transition for calculus students. Our intention was that by previewing many calculus questions in a rate of change (but merely algebraic) context, students would make connections between calculus rate of change problems, and their contextual knowledge and intuition.

We chose census data as an example of a non-physics context for change over time that would be familiar to our students. We asked students to make calculations and answer questions that were related to calculus questions they would see later in the semester. For example:

Census questions	Traditional calculus questions
Calculate a rate of change	Calculate a slope or derivative
Identify units for rate of change	Identify the dependent and independent variable in a derivative
Compare a rate of change graph to the census graph	Compare a derivative graph to the graph of the original function

The lesson follows a concrete to abstract progression that asks open-ended questions at several points. In particular:

- The columns in the table that students fill in have no horizontal lines. We designed it this way to see how students would line up the numbers with the previous columns. We found that if not prompted to think about it, most students would line up with the first two columns as in a spreadsheet table, and sometimes that led to errors in the first line. Our final version of the worksheet also has no horizontal lines, but we added a prompt for students to use deliberate reasoning to decide how to position these two columns. This prompts a richer discussion of how the initial values and the rates of change are related, and the issue is revisited in problem 5.
- In problem 5 students are asked to supply an appropriate representation for rate of change, with a prompt follow-up class discussion that should address how these are similar to derivative concepts and representations. The representations that students use often show the extent to which students are ready to think abstractly about rate of change.

- Problems 6-9 ask students to explain in words some things they have seen before, but may not have had to explain. These answers help to show the depth of student understanding.
- Problems 10-12 ask for students to draw conclusions based on the rate of change, which are accessible, but new for many students.

Related Files: Attached files relating to the lesson design are:
Rate_of_change.doc The lesson, including student pages and teachers notes
Rate_assignments.doc Assignments/extension activities to accompany the lesson
Rate_samples.doc Samples of student work including typical errors
Rate_revised.doc A revised version of the lesson including two new problems suggested by our observations the second time the lesson was taught.
UWRFCalc.mov Some clips from the first time the lesson was taught

PART III: THE STUDY

Introduction:

We chose the topic of rate of change, because in our experience, the process of moving from a single way of looking at derivatives (slope of a tangent line) to a broader contextual way of looking at derivatives (instantaneous rate of change) is often a difficult transition for calculus students. Our intention was that by previewing many calculus questions in a rate of change (but merely algebraic) context, students would make connections between calculus rate of change problems, and their contextual knowledge and intuition.

Approach:

We tried a lot of different things, because we weren't sure what would work best. We had a rubric of general things to look for [more later], we listened in on groups, had the lesson video taped, and asked the students to write their answers to the questions as well as sharing them orally (we collected and kept the group work from the sessions). The most valuable things we did were to just listen for things that were surprising, and to ask students to show us on paper what they were doing. (The videos were less useful than we would have expected because the sound quality was very poor especially when picking up student discussions. Possibly if one of us had felt competent to man the camera, we might have gotten more useful video documentation).

We learned most from the mistakes, and it's hard to predict what those will be in advance. Having noted those mistakes, we then were able to tweak some parts of the lesson to make the mistakes less likely, and we found that knowing what to look for helped us to identify more quickly which groups were making errors. We also found that students were making connections to calculus beyond what we specifically asked for, which suggests the addition of a question to probe specifically for those connections from the whole class (see **Conclusions** section). So the things that were most interesting were found almost by accident—by listening carefully to discussions, and by looking carefully at the written work.

Findings:

We discussed our in-person observations, and carefully crosschecked all of the written student work. We identified several anomalous answers that students gave, and discussed the thinking behind those answers. In follow-up activities with these and other classes we looked for similar errors, and determined that the following errors are both common enough and significant enough (reflecting a conceptual misunderstanding) that it is valuable for a teacher to be aware of, and watch for these errors in student work.

- In calculating rate of change from two ordered pairs, students may confuse rate of change with percent change when reporting their answers (students do not compute percent change—they compute rate of change, but then write it with a percent symbol, sometimes moving the decimal place first).
- In calculating and graphing rate of change from function data in a table, students may not recognize what to do with the first or last point in the table. Most often this shows up by students adding an extra line that should not be there to the change and rate of change columns (usually with 0 recorded as the change or rate of change), or with students adding an extra point to the graph (usually at 0). Students may make this error in the graph without making the similar error in the table.
- In calculating change or rate of change from function data given in a table, some students will calculate change from the first line of data in the table, even if the instructions specify that change should be computed between consecutive lines of data.

Additional minor students errors are discussed in the teachers' notes. We note that most of the significant errors that students make are in the parts of the activity that might be considered review of pre-calculus material. When we revised the lesson, we changed from having students graph the population data to providing a pre-made graph of population data, but the fact that students were making conceptual errors in the other more routine material made it clear that it was important to keep those routine calculations in the activity and that reviewing this material is valuable in clarifying student understandings of these background concepts.

Conclusions:

Students are asked to do, and are generally successful with, the specific learning goals of the lesson:
In the context of rate of change of population, students

- Explain the important differences between change and rate of change in the discussion about how to compute those values in the table, and also in their answer to question 9 (comparing the rates of change between two values whose net changes are nearly equal.)
- Identify the geometric interpretations of change in population, change in time, and rate of change of population in the geometric questions on page 3.
- Identify appropriate units for rate of change in context in problem 5 and the associated class discussion.
- Explain the contextual behavior associated with different rates of change in the problem (#10) where they discuss historical reasons for interesting rate of change values.

In the lesson, students move between tabular and graphical representations of data, and discuss how rates of change are influenced by changes in the real world context for the data. In short, students succeeded in all of the specific components that we wanted them to know, and that we believed would deepen their understanding of rate of change as a generalization of slope. Because there are so many parts of the course that contribute to students understanding of rate of change, we were not able to measure the impact of the lesson in a global sense. One of the teachers who used this lesson found that it became an important example to refer back to in later lessons, which argues both for its relevance to the curriculum, and makes it more difficult to separate out the impact of the lesson from the overall strategy of the course.

Suggestions for future development: Overheard conversations in student groups indicate that many students at this point in the class are more sophisticated mathematically than we are giving them credit for. Thus, we suggest replacing the teacher work of identifying the calculations algebraically by a question and discussion of the algebraic representations. Additionally, we would like to see a prompt added to encourage students to discuss and explicitly make those connections between the activity, and the calculus concepts they have been studying. These suggestions are incorporated in the file rate_of_change_revised.doc as problems #4 and #14.

Our experience indicates that the best observations came from the trained observers (ourselves), and the lesson as captured on camera was largely useless, as it was not effective at catching group discussions (too little of any given discussion was recorded, and the sound quality of the recordings was very bad). We recommend that groups attempting to capture group work in math classes consider having one of the lesson study teachers man the camera, as they may be more likely to make useful choices in what to focus on.

References:

Cerbin, Bill and Kopp, Brian (2005) *Using Lesson Study to Improve Teaching and Learning.* www.uwlax.edu/sotl/lsp.

Data and Story Library. *Predicting Appliance Sales.* http://lib.stat.cmu.edu/DASL/Datafiles/appliancedat.html.

GDP per Capita 1950 -. Statistics Sweden. http://www.scb.se/templates/tableOrChart____26670.asp

The 2007 Statistical Abstract: The National Data Book. (2001) US Census Bureau. http://www.census.gov/compendia/statab/population/

Related Files:

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UWRFCalc.mov Some clips from the first time the lesson was taught