

# Lesson Study in Calculus: Rolle's Theorem and the Mean Value Theorem

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## PART I: BACKGROUND

**Title:** Rolle's Theorem and the Mean Value Theorem

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**Discipline or Field:** Mathematics

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**Course Name:** Calculus I

### **Course Description:**

Calculus I (MATH 153) is the first course in a sequence of calculus courses required for students in various majors including Computer Science, Engineering, Mathematics, Science, and Technology. The main objective of the course is building the essential skills, mastery, and understanding of the applications of several topics including analytic geometry of plane functions, limits, continuity, derivatives of functions and applications; exponential, logarithmic, trigonometric and inverse functions; indefinite and definite integrals; and the fundamental theorem of calculus.

The prerequisite of this course is the completion of MATH 121, a trigonometric functions course, or an appropriate placement test score. The class size has increased over the last few semesters and averages about 40-45 students per class during the fall semester for MATH 153. During the spring semester when Calculus I is off sequence, the class size decreases.

All the students in Calculus I have laptops and graphing calculators, although these technologies are not crucial to this lesson. The classroom setting varies among the sections of the course from long tables to individual desks, making it difficult to generalize student interactions in the lesson plan. The lesson follows content on finding critical numbers of a function and locating relative extrema on a closed interval. It includes two major theorems about continuity and derivatives of functions, which also have several applications, and the lesson usually takes approximately two days to complete.

The students are split depending on their major into the two versions of the course. Calculus and Analytic Geometry I (MATH 156) contains mainly students majoring in Computer Science and Mathematics, and MATH 153 contains all other majors. The two versions are almost identical with the version for the Computer Science and Mathematics majors requiring more proofs and theory. Although this lesson was tested in a MATH 153 classroom, it could easily be used for MATH 156 or any calculus course including the topics of Rolle's Theorem and the Mean Value Theorem.

### **Executive Summary:**

The topic of the lesson is Rolle's Theorem and the Mean Value Theorem.

#### *Learning Goals.*

1. Students will understand the meaning of Rolle's Theorem and the Mean Value Theorem, including why each hypothesis is necessary.
2. Students will complete problems and applications using Rolle's Theorem and the Mean Value Theorem.
3. Students will appreciate the discovery process of developing mathematics and have a better understanding of the construction and proof of mathematical theorems.

*Lesson Design.* The lesson was designed in order to emphasize the discovery process and the role of proof in mathematics. The first major portion of the lesson is an activity that asks students, in several steps, to draw graphs of functions satisfying various hypotheses. The last graph that students are asked to draw is impossible to draw, because any graph satisfying all of the required conditions would violate Rolle's Theorem. Rolle's Theorem is introduced in this way. A second activity involving graphs related to the Mean Value Theorem is used to introduce or study the this theorem. These graphing exercises are intended to help students discover for themselves the two theorems and help them to appreciate the discovery process in mathematics.

The second major portion of the lesson is to work problems involving the theorems to better understand how the theorems are used and apply in practice. The variety of problems is intended to emphasize different aspects of the theorems, including why the hypotheses are necessary and how to apply the theorems to modeling applications and more abstract settings.

The final portion of the lesson is to prove the Mean Value Theorem assuming Rolle's Theorem. This portion of the lesson is expected to be difficult for students, so ample time should be allotted for question and discussion.

*Major Findings.* During the first round of the lesson, we learned that students seem to catch on quickly that the second graphing exercise is almost identical to the first and therefore the last graph is impossible to draw. This seemed to cause a significant reduction in their engagement with the lesson. However, when this activity was changed for the second round, the decrease in performance on problems involving the hypotheses of the Mean Value Theorem suggests that the repetition may actually have served its purpose of emphasizing the hypotheses present in the theorem.

Another interesting observation we found during the first round of the lesson is that students seemed unsure of which aspects of the lesson they were having trouble with. Survey results indicated that many students thought application problems were the most confusing part of the lesson, but the scores on such homework problems suggest that solving application problems was among students' strongest points.

## PART II: THE LESSON

### How to Teach the Lesson

In preparation for teaching this lesson, the instructor makes copies of the handouts to be used. These includes several worksheets on which the students will draw graphs in class (the first of these is given in the appendix, the others are very similar), as well as a set of homework assignments, a quiz, and a student evaluation of the lesson. All of these documents are in the appendix.

The instructor also prepares to give statements of Rolle's Theorem and the Mean Value Theorem, as well as a proof of the Mean Value Theorem using Rolle's Theorem. He/she also prepares to work through example problems with the class using both theorems. (See In-Class Examples in the appendix.) This process may take up to an hour of preparation time.

An outline of the following is included in the appendix:

The instructor begins the first day of the lesson by writing the following problem on the board intended to inspire the students to learn the material: Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they had the same speed. This problem is then eventually given as homework problem #5.

The instructor asks the class to draw at least four graphs of functions,  $f(x)$ , on their worksheets that are continuous on  $[2,7]$  with  $f(2) = f(7) = 1$ . The instructor further asks the class to make sure that there is no point on their graph where  $f(x)$  has a horizontal tangent line. As the students make their attempts, the instructor can observe and give feedback to the students as to what they are drawing. Eventually, the students are asked to compare with their neighbors to arrive at a correct solution. The instructor could draw examples on the board that have a horizontal tangent line to illustrate what the students, in their graphs, are trying to avoid. When all the students have completed the exercise, the instructor could share on the board some correct solutions. The goal of this exercise is to stress that there must be a point at which the function is not differentiable.

The instructor then asks the students to add a condition and repeat the graphing exercise: Make sure the function is differentiable on the interval  $(2,7)$ . Once again he/she asks the students for four graphs, to encourage them to make multiple attempts. Feedback during the exercise can be an important part of the

process, as can discussion and comparison between the students. The fact that no graph can be drawn that has no horizontal tangent line once the condition of differentiability is added leads directly to the statement of Rolle's Theorem, and the two graphing exercises have pointed out the necessity of differentiability and continuity in the hypotheses of the theorem.

The instructor gives the statement of Rolle's Theorem on the board. He/she may give graphical examples to show that there may be more than one horizontal tangent line. He/she may reiterate the necessity of differentiability in the hypotheses by referring to the first graphing exercise. Afterwards, a simple application problem can be worked on the board with the students. For example: Given  $f(x) = x^2 - x + 8$ , explain why we know that there is a  $c$  in  $(-2, 9)$  with  $f'(c) = 0$ . Find the value  $c$ .

In the first version of the lesson taught, it continued with another graphing exercise essentially identical to the Rolle's Theorem exercise except that the graphs were required to start and end at different  $y$ -values and instead of not having a horizontal tangent line, they were not supposed to have a tangent line parallel to the secant line between the starting and ending points. For the second version of the lesson, this activity was modified. The second version of the lesson continues with the instructor simply asking the class to draw four different graphs of functions,  $f(x)$ , that are differentiable on  $(2, 7)$ , continuous on  $[2, 7]$ , and such that  $f(2) = 1$  while  $f(7) = 6$ . The instructor may once again offer feedback during the graphing process, as well as encourage collaboration. It should be stressed, though, that each student should make four different graphs of their own.

The instructor then initiates another class discussion: What do all of your graphs have in common? Do they all have horizontal tangent lines? If so, could one be drawn without a horizontal tangent line? The instructor should encourage students to try this; room is given on the worksheet for a total of six graphs for each exercise. What about the secant line? Can a graph be drawn without a tangent line that is parallel to the secant line? The instructor could reinforce this concept by drawing the secant line on a graphic example of a function meeting the above criteria. This discussion should lead to the statement of the Mean Value Theorem.

The instructor writes the Mean Value Theorem on the board. He/she may give graphical examples to show that there may be more than one tangent line parallel to the secant line. He/she may reinforce the similarities between Rolle's Theorem and the Mean Value Theorem by showing the necessity, once again, of differentiability in the hypotheses. This could be done by illustrating an example where the function is not differentiable and does not have a tangent line parallel to the secant. Afterwards an application problem can be worked on the board with the students. For example: Verify that  $f(x) = \sqrt{x}$  satisfies the hypotheses of the Mean Value Theorem on  $[0, 9]$  and find the value  $x = c$  guaranteed by it.

As time permits on the first day, the instructor can include more applications and problems. See the In-Class Examples in the appendix. Ideally, the problem proving that a given function has only one root (In-Class examples #3) should be worked as a proof by contradiction, because this gives the students some exposure to the idea of proof before the next day of the lesson, where a complete proof of the Mean Value Theorem will be given.

At some point during day one, the instructor collects all the worksheets that have been used to do the various graphing exercises. Note that this was done as part of our observation and evaluation of the lesson. In general, it is the instructor's decision whether or not to collect the students' graphs.

To start the second day of the lesson, the instructor helps the students recall the two theorems. He/she may draw graphs depicting both theorems. He/she may help the students see that a graph for Rolle's Theorem may simply be 'tilted' a bit to depict a graph for the Mean Value Theorem. He/she may put the statements of both theorems somewhere on the board as a reference for the students.

Significant class time may then be devoted to helping the students work through a proof of the Mean Value Theorem using Rolle's Theorem. The idea is that the students see both the structure of a proof and the connection between Rolle's Theorem and the Mean Value Theorem. Although the instructor will be providing the bulk of the information needed to complete the proof, the class can be asked to provide much information needed in each step of the proof to complete it. Significant discussion can be made at each step to insure that the students become confident in their abilities to understand the design of a proof and to insure that they gain some confidence in their own abilities to construct even the smallest of steps involved.

Afterwards, remaining class time can be devoted to applications, problems, and examples. A straightforward Mean Value Theorem problem could be worked, where the hypotheses are met, and the exercise is to find the guaranteed  $x = c$  value. A problem could be presented where Rolle's Theorem does not apply. (i.e., there is no  $x = c$ ) and the exercise is to determine that the condition of differentiability is not met and where it is not met. See In-Class Example #1 in the appendix.

A sheet of homework assignments should be handed out at the end of the second day of the lesson. See the Homework set in the appendix. The students should have a few days to work on these assignments (over the weekend, for example). These should be handed in and graded in order to gauge student understanding and the effectiveness of the lesson.

After the students have had time to do the homework set, a third lesson day could include a quiz (see the Quiz in the appendix) and a student evaluation of the lesson (see the Post-Lesson Student Survey in the appendix). Collecting the evaluations on the same day they are handed out works best, as we experienced difficulties collecting them at a later time. The quiz, too, should be handed in and graded, again as a measure of student understanding and the effectiveness of the lesson.

### **Student Learning Goals**

The topic of Rolle's Theorem and the Mean Value Theorem was chosen for several reasons. First of all, they are two major results related to derivatives, and they are strongly connected. In fact, the Mean Value Theorem is essentially a generalization of Rolle's Theorem. Often, students see these theorems in a course but do not really understand the results and why the hypotheses are useful. From a broader view, it is important for students to understand how mathematical theorems are constructed, proven, and then applied to solving problems. This lesson is well-suited for those goals.

The student learning goals for the lesson were the following:

1. Students will understand the meaning of Rolle's Theorem and the Mean Value Theorem, including why each hypothesis is necessary.
2. Students will complete problems and applications using Rolle's Theorem and the Mean Value Theorem.
3. Students will appreciate the discovery process of developing mathematics and have a better understanding of the construction and proof of mathematical theorems.

The first goal was developed to ensure students understand the basis of the theorems and why the given assumptions are necessary to achieve the conclusions of the theorems. This requires that students' utilize and/or develop critical thinking skills to be able to comprehend the theorems and their meanings. The purpose of the second goal is to have students apply the theorems to basic problems and applications, hopefully connecting the theorems to their chosen discipline. Finally, the third goal is a more general, long-term goal in which students will appreciate how mathematicians discover and prove results. The hope is that students will appreciate this discovery process and use it in future courses.

### **How the Lesson is Intended to Work**

Moving from the lesson on paper to lesson in practice offers lessons in itself. The instructor begins the first day of the lesson by writing a problem on the board intended to inspire the students to learn the material. This activity is design to open up the discussion about the topic and also to point out the practicality and applicability of the topic. The following two activities are used to lead to the hypothesis in the statement of the Roll's theorem. The idea is that if students discover the conditions on their own it is going to be easier to remember them when solving problems.

Next, the instructor introduces the statement of Rolle's Theorem and a follow up example. By giving the statement of Rolle's Theorem on the board, the instructor reiterates the importance of the hypothesis conditions.

This next activity is intended to lead, using a discovery approach, to the statement of the Mean Value Theorem. Following the statement of Mean Value Theorem, one example is used to reinforce the similarities between Rolle's Theorem and the Mean Value Theorem.

The design of the lesson is done with the following idea in mind: that deep learning occurs when meaningful connections with the content is made, when the purpose for learning is clear, when the environment supports critical learning, and when there is mutual engagement in the learning process between teacher and students, and between student and student. We think that the discovery process used was essential to assist students in learning.

In designing the lesson, we try to use, whenever possible, examples that were specific to our student population (see examples 2 and 5 in the appendix). We think that this plays a role in making the lesson meaningful to our students.

## **PART III: THE STUDY**

### **Approach**

Our study of the lesson involved collecting data from four different types of sources: observation of the lesson during class by team members, outside observers, and videotape; samples of students' in-class work in the form of worksheets that they handed in after class; grades on homework, quiz and exam questions; and a short student survey administered during the period following the lesson.

All of the observers for this lesson were members of the Mathematics, Statistics, and Computer Science Department at UW-Stout, but not all were members of the lesson study team. They were asked to place themselves discretely in the room and record their observations on a form that we provided. In order to keep the observers focused on the lesson itself and the students rather than the style and performance of the actual teacher, we provided each observer with an observation form. The observation form appears below in the appendix. For each section of the lesson, this form contained focal questions and aspects of the lesson that we wanted the observers to focus on. Additionally, the following four focal questions appeared at the top of the form, and were intended to guide the general observations.

*During lecture/discussion time:*

Do students seem engaged? (Participating, note taking, asking questions, visual facial cues (nodding, etc))

How many students are engaged? Who is engaged?

Who is actively participating?

*During exploration time (graphs):*

Do students understand what they are trying to draw?

Where are they getting stuck?

How did they get unstuck? (Did an instructor help them, another student, etc?)

Did the students feel they got the right picture at the end?

The in-class work we collected consisted of the worksheets and papers on which they drew the graphs requested in the graphing section of the lesson. Students were also provided with extra graphing worksheets on which they could copy the graphs that they had to turn in or take notes for their own use.

During the first round of the lesson, we recorded scores on particular homework and quiz problems that covered material from the lesson. In order to determine how well students retained the content, we additionally paired each student's quiz with his or her homework. During the second round of the lesson we attempted to assess the difference between students' responses to our lesson and a "non-lesson study" lesson by putting a common problem on the final exam in two sections of this class: the lesson study section and one section that did not participate in the lesson study. We recorded average scores on this problem and compared the two.

The student survey was distributed the day after the lesson and collected that day during the first iteration and several days later in the second iteration. It was intended to gauge students' engagement in the lesson, as well as determine the parts of the lesson that they found difficult or frustrating. The complete survey appears in the appendix of this report.

### **Findings**

Observer Comments: After the lesson, the team reviewed the observers' comments and one team member summarized the comments in a 12-page summary. From the first iteration, there were three common observations. The first was that many of the students needed to be reminded of what discontinuities and points of non-differentiability look like in the graph of a function. Many students also seem to need reminding of what a secant line is and what one looks like on a graph. The second common observation was that during the graphing exercises, many of the students who got stuck on an aspect of the activity looked towards their neighbors for help in getting un-stuck, and this strategy was very effective for most students. The third common observation is that on the second day there was a noticeable loss of engagement during the proof of the Mean Value Theorem, but the participation level picked up again when the class moved on to the examples and exercises.

In-Class Work: After collecting the in-class worksheets, the team reviewed them looking for common themes. The number of papers displaying each of these themes is displayed in the following tables. Because the second graphing activity was changed for the second iteration of the lesson, we had to tabulate different themes. Since part of the goal of the graphing activities was to help students actually to discover Rolle's Theorem and the Mean Value Theorem, we were interested in both where students were going wrong and whether or not they eventually did seem to discover the theorems. Therefore, for the first iteration of the lesson, we counted the number of students who indicated that the last graphing exercise in both Rolle's Theorem and the Mean Value Theorem was impossible, as it was.

**Table 1; Role's Theorem Graph Data, Continuity Activity**

Number of graphs drawn					
	1 Graph	2 Graphs	3 Graphs	4 Graphs	5 Graphs
<b>Round 1</b>	27 (63%)	12 (28%)	3 (7%)	1 (2%)	0 (0%)
<b>Round 2</b>	4 (11%)	7 (20%)	7 (20%)	13 (37%)	4 (11%)
Correct/Incorrect					
Correct graph present			No correct graph present		
<b>Round 1: 41</b>	<b>Round 2: 35</b>		<b>Round 1: 2</b>		<b>Round 2: 0</b>
Common Errors					
Error	<b>Round 1</b>		<b>Round 2</b>		
not a function	2		3		
discontinuous	2		0		
horizontal tangent	18		15		

**Table 2; Role's Theorem Graph Data, Differentiability Activity**

Number of graphs drawn						
	0 Graphs	1 Graph	2 Graphs	3 Graphs	4 Graphs	5 Graphs
<b>Round 1</b>	4 (10%)	22 (54%)	11 (27%)	3 (7%)	0 (0%)	1 (2%)
<b>Round 2</b>	26 (25 with 2 dots only) (74%)	2 (6%)	5 (14%)	1 (3%)	1 (3%)	0 (0%)
Correct/Incorrect						
Number saying "impossible"			<b>Round 1: 20</b>		<b>Round 2: 6</b>	
Common Errors						
Error	<b>Round 1</b>			<b>Round 2</b>		
horizontal tangent	16			14		
Discontinuous	11			0		
not differentiable (but continuous)	8			0		
not a function	4			0		
wrong endpoints	3			0		

**Table 3; Mean Value Theorem Graph Data, Continuity Activity, Round 1**

Number of graphs drawn			
Graphs	1	2	3
Papers	16	19	8
Correct/Incorrect			
Correct graph present		No correct graph present	

43	0
Common Errors	
Error	Number
parallel tangent (but not a straight line)	16
not a function	5
straight line	4
crossed off correct answer	1

**Table 4; Mean Value Theorem Graph Data, Differentiability Activity, Round 1**

Number of graphs drawn				
Graphs	0	1	2	3
Papers	6	31	3	3
Correct/Incorrect				
Number saying “impossible”		30		
Common Errors				
Error	Number			
parallel tangent (but not a straight line)	11			
not differentiable (but continuous)	6			
discontinuous	4			
not a function	5			
straight line	1			

**Table 5; Mean Value Theorem Data, Round 2:**

Item	Number
At least 2 different graph shapes drawn	35
Drew the secant line at least once	15
Drew in a tangent line parallel to the secant line at least once	20
At least one graph without a horizontal tangent line	20
Total number of paper	35

General observations from the first day of the lesson were that students were actively engaged during the hands-on graphing examples. The worksheets on which the students graphed their examples and counterexamples during the graphing activities were arranged in such a way as to allow for the collection of data on the students’ work and progress towards solutions. The most common error for the continuity activity with Rolle’s Theorem was having a horizontal tangent line, but in the end, 41 of the 43 students had a correct graph. For the differentiable graphs in the Rolle’s Theorem activity, the most common errors were graphs that had horizontal tangent lines, were not continuous, or were not differentiable. At least half of the students figured out that it was impossible to draw a differentiable function without a horizontal tangent line and meeting the given conditions. Another striking observation from the data is that for the Rolle’s Theorem continuity exercise, 39 of 43 papers contained two or fewer attempted graphs, and for the differentiable exercise, 37 of 41 contained two or fewer graphs. These data were the main reason that we decided to change the Mean Value Theorem graphing exercise in the second iteration of the lesson. We wanted to give students a new kind of graphing exercise that would hopefully keep their attention more strongly than essentially repeating the previous exercise with different hypotheses.

In the Mean Value Theorem graphing activity of the first iteration, many students had trouble identifying a tangent line parallel to the secant line. The most common error in graphs for the continuity activity with the Mean Value Theorem was to have a point where the tangent line was parallel to the secant line. For the differentiable activity, the most common errors were graphs that had tangent lines parallel to the secant line, were not continuous, or were not differentiable, as in the Rolle’s Theorem activity. In the end, almost three-quarters of the students figured out that it was impossible to draw a differentiable function satisfying the given conditions.

Homework and Quiz Results: The average scores for each homework and quiz problem assigned during the lesson are presented in Table 6. The actual question corresponding to the problems referenced in the tables can be found in the appendix. All of the homework and quiz problems were traditional hand-written assignments, and students were expected to show all of the work involved in solving the problems.

**Table 6; Quiz and Homework Scores, Round 1**

	Quiz				Homework							
	Question 1		Question 2	Quiz Total	Question 1		Question 2		Question 3	Question 4	Question 5	Homework Total
	a	b			a	b	a	b				
Average Round 1	4.40	2.38	2.40	9.19	1.83	1.88	1.88	1.49	1.69	3.21	3.64	13.41
Average Round 2	3.20	3.27	2.93	9.25	2.00	1.91	1.56	1.12	2.88	2.77	3.75	15.14
Round 2 – Round 1	-1.20	0.89	0.53	0.06	0.17	0.03	-0.32	-0.37	1.19	-0.44	0.12	1.73
Percent Change	-24.00	17.80	10.60	0.40	8.50	1.50	-16.00	-18.50	29.75	-11.00	3.00	8.65
Total	5	5	5	15	2	2	2	2	4	4	4	20

To assess how well the lesson achieved Goal 2 from Section 2, two difficult application problems, numbers 4 and 5 shown below for convenience, were assigned.

- Show that the equation  $1 + 2x + x^3 + 4x^5 = 0$  has exactly one real root.
- Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed. (Hint: Consider  $f(t) = g(t) - h(t)$ , where  $g$  and  $h$  are the position functions of the two runners.)

The average score on these two problems was about 3.4 out of 4 points in the first iteration and 3.3 out of 4 in the second iteration, which suggests that students understand the applications of the theorems enough to successfully work problems, and that Goal 2 of Section 2 was at least partially met. This contradicts the students' responses in the survey in the first round in which nine of 40 students indicated problems, examples, or applications were the most confusing aspect of the lesson, with an additional two identifying homework as most confusing. Perhaps their responses are measuring the difficulty they had in achieving understanding of the problems and concepts, rather than their final level of understanding.

Some of the results and student survey responses suggest aspects of the lesson which could be improved. In particular, in Round 1 students averaged 1.7 out of 4 points on homework question 3:

- Let  $f(x) = (x - 1)^{-2}$ . Show that  $f(0) = f(2)$  but there is no number  $c$  in  $(0, 2)$  such that  $f'(c) = 0$ . Why does this not contradict Rolle's Theorem?

This average score indicates that many students did not understand the need for or role of the hypotheses in these theorems. Students were not connecting the counterexamples they drew in the non-differentiable graphing activities with the fact that differentiability is necessary for the conclusions of the theorems to hold. The average score of 2.9 out of 4 in the second round seems to indicate that students in the second may have acquired a slightly stronger understanding of the role of hypothesis in mathematics.

Exam Results: For the second round of the lesson, the final exam problem included in the appendix was placed on the final exam in the lesson study section and in a non-lesson study section. The sections' average scores out of 25 points are shown in the following table and the raw data for the two sections in the following stem and leaf plots. The exam question was a traditional hand-written problem for which students were expected to show all of their work. The instructors of the two sections agreed in advance on a rubric for grading the problem, but their applications of the rubric may have differed. Finally, this data does not take into account the total weight of the final exam in the courses or any final scaling or "curving" of the final course grades.

**Figure 1; Final Exam Question Raw Data**      1:6 = 16 points

Lesson Study Section

Non-Lesson Study Section

0 : 0023  
 0 : 5557778  
 1 : 000000000000333444  
 1 : 55666  
 2 : 0003  
 2 : 55

0 : 00000000000000  
 0 : 55555555558  
 1 : 0000000000000000  
 1 : 58  
 2 :  
 2 : 5

**Table 7; Final Exam Question Scores, Round 2**

	Average out of 25	Standard Deviation
Lesson Study Section	11.525	6.079
Non-Lesson Study Section	6.413	5.459

Survey Results: The student surveys contained three questions, and the following tables show the most common responses to each question during both iterations of the lesson. During the first iteration of the lesson, almost all of the students present completed and turned a survey form, and the results of this survey are shown below. However, for the second round, only 7 out of 40 students returned a completed form, possibly because students were allowed to take the survey home and return it later. Because of the extremely low response rate in the second section, we elected not to analyze the survey data from the second section, and we strongly suggest that any survey data collected be collected during class time.

**Table 8; Most Common Survey Responses, Round 1**

<b>What part of the lesson was most engaging? Why?</b> (40 responses)	
sketching activities	15
Rolle's Theorem	4
homework/problems	3
<b>What part of the lesson was most confusing? Why?</b> (40 responses)	
problems/examples/applications	9
proof/second day	5
Homework	2
<b>Please provide any additional comments you have regarding the lesson.</b> (20 responses)	
slow down/more examples	5
general unhappiness/I don't like math	6

Student responses from the surveys indicate that students enjoyed the graphing activity and found the visualization helpful. In fact, 15 out of 40 students who responded to the survey in the first round specifically mentioned the graphing activity as the part of the lesson that was most engaging. Therefore, the lesson did seem to successfully achieve Goal 3. However, for both theorems, observers' comments suggest that the transition from the graphing activity to note-taking for the discussion of the formal statements of the theorems was difficult for students. Since understanding the formal theorems and their hypotheses is part of the Goal 1 for the lesson, a revision of this lesson should include a better transition from the intuitive discovery process to the formal statements of the theorems. Overall, students seemed to appreciate and at least to be engaged with the discovery process.

**Discussion**

One of our goals for the lesson was that students will understand the meaning of Rolle's Theorem and the Mean Value Theorem, including why each hypothesis is necessary. In our first iteration of the lesson, the graphing exercises for both Rolle's Theorem and the Mean Value Theorem stressed the need for the hypotheses and the fact that removing one of the hypotheses allows for examples where the conclusions of the theorems do not hold. Since students seemed to catch on quickly in the first iteration of the lesson that the last Mean Value Theorem graphing exercise, was impossible, we chose to modify this activity in the second iteration in order to keep students more engaged. The new activity focused more on the conclusion of the theorem and less on the hypotheses. This was the largest change between the two iterations. However, its effect on student learning was very difficult to judge from the quiz and homework scores collected.

For the sake of consistency in the data collected during the study, the homework and quiz questions were not modified at all between the two iterations of the lesson. Two trends are apparent in the homework and quiz score data from the two iterations of the lesson. The first is that in both questions (homework #2a and quiz #1a) dealing with checking the hypothesis of the MVT, there were large decreases of 16% and 24% respectively between the first and second iterations. This suggests that the modified lesson was less effective at helping students understand this part of the theory than the first draft was. This seems to make sense, because the portion of the graphing exercise relating to the hypotheses of the MVT was decreased from the first iteration to the second. This suggests that if students' ability to check the hypotheses of Rolle's Theorem and the Mean Value Theorem is important to the instructor teaching the course, it may be better to use the lesson from the first round in which students were asked to repeat essentially the same activity that they did in the setting of Rolle's Theorem in the setting of the Mean Value Theorem or a slight modification of this activity. This conclusion is supported also by the fact that the graphing activity relating to Rolle's Theorem was not changed between the two iterations and on the single question (Homework #1a) asking students to check the hypotheses of Rolle's Theorem, students in round 2 scored 8.5% higher than students in round 1.

Contrasting with the possibly decreased ability to check the hypotheses of the MVT is the fact that on homework question #3, which asks students to explain why a certain example not satisfying the hypotheses of the MVT does not actually contradict the MVT, students in the second iteration scored 29.75% higher than students in the first iteration. This suggests that the even though the students from iteration 2 may be more confused than their peers from iteration 1 about exactly what the hypotheses of the MVT are, they are clearer on the abstract notion of the necessity of the hypotheses of a theorem and the fact that a theorem does not apply to an example that does not satisfy its hypotheses. While this may be true, it is not clear to us which part of the revised lesson, if any, contributed to this increased understanding.

The last differences between the iterations that were in the 20% range were a decrease of 16% on question #2a of the homework and an increase of 17.8% on question #1b of the quiz. These two questions were very similar to each other, both dealing with finding the value or values of  $c$  guaranteed by the Mean Value Theorem. Additionally, the same professor graded all of the homework and all of the quizzes for both iterations of the lesson, so it is unlikely that a difference in grading style is the cause of these major differences in score. Therefore, the large increase in the scores on one question and the large decrease in scores on the other question are very confusing. We currently have no satisfactory explanation for this difference.

The final observation we have made from the quiz and homework scores is a comparison of student performance to the results of the survey. We made this comparison only for the first round, since we did not have reliable survey data for the second round. The average score on the two homework problems dealing with applications, numbers 4 and 5, was about 3.4 out of 4, which suggests that students understand the applications of the theorems enough to successfully work problems, and that Goal 2 of was at least partially met. This contradicts the students' responses in the survey in which nine of 40 students indicated problems, examples, or applications were the most confusing aspect of the lesson, with an additional two identifying homework as most confusing. Perhaps their responses are measuring the difficulty they had in achieving understanding of the problems and concepts, rather than their final level of understanding.

During the study, we made one comparison between a section of the course that received the lesson study lesson and one that did not. The comparison we made was that during round 2, we put a common final exam problem dealing with the Mean Value Theorem on the final exam in the lesson study section and in one section that did not receive the lesson study lesson. As shown above in Table 7 and Figure 1, there was a large difference in the average scores between the two sections and a large difference in the shapes of the distributions of scores in the sections. There exist several explanations for these differences. The first is that the grading styles of the two instructors for the two sections were different. This conclusion is supported by the difference in the shapes of the distributions of the scores in the two classes. Another explanation of the increased average score in the lesson study section is that class periods of the lesson were more memorable in the lesson study section due to the presence of the video camera and outside observers. This might have caused the students in that section to surmise correctly that this is a topic that would figure strongly on the final exam and that they therefore should study it particularly carefully during their preparation for the final exam. A third explanation for the difference in average scores is simply that the lesson study produced a very memorable and pedagogically sound lesson. Overall, we believe that a combination of these three factors probably caused the differences, and with only two sections to compare, we are unable to draw a conclusion with confidence regarding the most important cause of the differences.

Another goal of the lesson was that students will appreciate the discovery process of developing mathematics. This goal seems to have been met equally well in both iterations. For the first iteration, 15 of 40 students who responded to the survey indicated that the graphing activities were the most engaging part of the lesson. Therefore, students seem to be interested in the activities in which they discover the statements of the theorems for themselves. On the other hand, in both iterations, many students still had a lot of trouble with theoretical questions that asked them to apply the knowledge that they discovered during their graphing exercises. This is indicated in the relatively lower scores on homework problem 4 than on the rest of the less theoretical homework problems in both rounds, especially in iteration 2. This seems to suggest that students still have a lot of trouble deeply understanding what they discover during the graphing exercises and applying it to new situations.

After analyzing the data from the second iteration of the lesson study process on this lesson, we can suggest several changes from the second iteration. First, we would suggest moving the problem asking students to prove that a certain equation has only one real zero to the first day. This would introduce the idea of proof the first day and hopefully prepare students for the proof of the Mean Value Theorem, which appears on the second day. If time allows, for example in a class period of 75 minutes, we suggest having at least one example on the first day of a function that fails to satisfy the hypotheses of one of the theorems and also fails to satisfy the conclusion of the theorem. This, we believe, would help students to better see the need for each hypothesis of the theorems. Secondly, Rolle's Theorem is a consequence of the Mean Value Theorem, but students have a lot of trouble seeing the relationship between the two theorems. Therefore, we suggest that a little time be spent on indicating that Rolles' Theorem is really the Mean Value Theorem applied to the situation in which  $f(a) = f(b)$ , because in this situation,  $(f(a) - f(b))/(b-a) = 0$ . Finally, the change between the two iterations in the Mean Value Theorem graphing activity seems to have had mixed results. The new activity puts less emphasis on the hypotheses of the theorem, but it is different enough from the activity that precedes it to keep the students attention more successfully than the Mean Value Theorem activity in the first round. This change seems to have resulted in a decrease in ability to verify the hypotheses of the Mean Value Theorem but an increased understanding of the abstract role of hypothesis in mathematical theorems. If possible, we suggest that an activity be considered that somehow both keeps students' attention by being different enough from the Rolle's Theorem activity but still emphasizes the particular hypotheses of the Mean Value Theorem enough for students to remember them as well as they do those of Rolle's Theorem.

## References

Thomas, G., Mauuece, W., Hass, J , and Giordano, F., *Thomas' Calculus, 11<sup>th</sup> edition*, Addison Wesley, USA, 2005.

## PART IV: APPENDIX

### Outline

This outline gives, briefly, the information contained under How to Teach the Lesson. It can be used as a reference for the instructor while preparing for the lesson, and during the lesson itself.

Day One:

- I. Today's topic: Two theorems that let us find information about  $f'$  from information about  $f$ .  
First let's experiment.
  - A. Introductory problem: Two runners start a race at the same time and finish in a tie.  
Prove that at some time during the race they had the same speed.
  - B. Draw at least 4 graphs of functions  $f(x)$  such that:
    - \*  $f(x)$  is continuous on  $[2,7]$
    - \*  $f(2) = 1, f(7) = 1$
    - \* there is no point where  $f(x)$  has a horizontal tangent line
    - \* (encourage students to find more than one possible "problem")
  - C. Compare with your neighbor
  - D. Discuss and/or help students draw one.

- \* horizontal tangent  $\Rightarrow f'(c) = 0$
  - \* is there a way to change direction without  $f' = 0$ ?
- E. Now try to draw  $f(x)$  such that
- \*  $f(x)$  is differentiable on  $(2,7)$ , continuous on  $[2,7]$
  - \*  $f(2) = 1, f(7) = 1$
  - \* no value  $c$  in  $(2,7)$  with  $f'(c) = 0$  (horizontal tangent line)
- F. Compare/discuss

II. Rolle's Theorem – state and discuss

- A. The theorem
- B. There may be more than one value with  $f'(c) = 0$ .
- C. Compare to pictures
- D. Reminder – you need differentiability!
- E. Very simple application problem. Something like  $f(x) = x^2 - x + 3$ .  
Explain why we know that there is a  $c$  in  $(-2,3)$  with  $f'(c) = 0$ . Find the  $c$ .

III. The Mean Value Theorem

- A. Now draw 4 different graphs of functions  $f(x)$  such that:
  - \*  $f(x)$  is differentiable on  $(2,7)$  and continuous on  $[2,7]$
  - \*  $f(2) = 1, f(7) = 6$
- B. What do you notice that all of your graphs have in common?
  - \* horizontal tangent line?
  - \* think about the secant line (draw one on the board)
  - \* think about parallel lines
  - \* do you think you could draw one without a tangent line parallel to the secant?
- B. The Theorem – State and Discuss
- C. Applications and problems
  - \* Verify that  $f(x) = \sqrt{x}$  satisfies the hypotheses of the MVT on  $[0,9]$  and find the value  $x=c$  guaranteed by the MVT

IV. (As time permits) Applications and problems

- \* If time permits, do the “only one root” problem as a proof. Ideally:
  - A. Menomonic to Madison (In-Class Example #5)
  - B. How large can  $f(a)$  be? (In-Class Example #4)
  - B. Only one root (In-Class Example #3)

Day Two:

I. Recall Rolle's Theorem and the MVT.

- A. What do they say? -- pictures
- B. Recall that the pictures are related
- C. The proof.

II. Applications, problems, examples – Pick up from IV on day one. Work in groups on the later problems, and present solutions to the class.

- A. Straightforward MVT problem, find  $c$ .
- B. Rolle's Theorem doesn't apply because of differentiability (In-Class Example #1)
- C. Speeding Car. (In-Class Example #2)
- D. Only one root. (In-Class Example #3)

III. Hand out homework set

Day Three (not necessarily the very next class period):

- I. Collect homework
- II. Quiz – collect!
- III. Student evaluations of the lesson (Post-Lesson Student Survey) – collect!

**In-Class Examples**

This is the list of examples during class while teaching this lesson. The problems were done on the board, with the instructor guiding the students through each problem.

1. Let  $f(x) = 1 - x^{\frac{2}{3}}$ . Show that  $f(-1) = f(1)$  but there is no number  $c$  in  $(-1, 1)$  such that  $f'(c) = 0$ . Why does this not contradict Rolle's Theorem?
2. Betty is driving through Menomonie. Two patrol cars with radar detectors are stationed 0.75 mile apart. She passes the first patrol car by Lake Menomin and is clocked at 30 miles per hour. One minute later, she passes another patrol car and her speed is clocked at 34 miles per hour. Show that somewhere between the two cars Betty exceeded the 35 mile per hour speed limit.
3. Show that the equation  $x^5 + 10x + 3 = 0$  has exactly one real root.
4. There is a certain unknown function  $f(x)$ . The only things that you know about  $f(x)$  is that it is differentiable on the entire real line  $(-\infty, \infty)$ , that  $f'(x) \leq 3$  for all  $x$ , and that  $f(2) = 5$ . What is the large possible value that  $f(8)$  can take on?
5. Madison is about 200 miles from Menomonie. The top speed of your car is 60 mph. Give an argument as to why you can be sure that you will be late for your meeting you have in 3 hours, but you are currently still in Menomonie.

### Homework

This is the set of homework we handed out to the students to work on by themselves outside of class. These were collected and graded.

1. Let  $f(x) = x^2 - 4x + 1$ .
  - a. Verify that  $f$  satisfies the three hypotheses of Rolle's Theorem on the interval  $[0, 4]$ .
  - b. Find all numbers  $c$  that are guaranteed by Rolle's Theorem.
2. Let  $f(x) = \frac{x}{x+2}$ .
  - a. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the interval  $[1, 4]$ .
  - b. Find all numbers  $c$  that satisfy the conclusion of the Mean Value Theorem.
3. Let  $f(x) = (x-1)^{-2}$ . Show that  $f(0) = f(2)$  but there is no number  $c$  in  $(0, 2)$  such that  $f'(c) = 0$ . Why does this not contradict Rolle's Theorem?
4. Show that the equation  $1 + 2x + x^3 + 4x^5 = 0$  has exactly one real root.
5. Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed.  
(Hint: Consider  $f(t) = g(t) - h(t)$ , where  $g$  and  $h$  are the position functions of the two runners.)

## Quiz

We gave this quiz after the students had had time to do the homework problems by themselves at home. It was collected and graded.

1. Let **Error! Bookmark not defined.**  $f(x) = x^3 - 2x$ .
  - (a) (5 points) Verify that  $f$  satisfies the two hypotheses of the Mean Value Theorem on the interval  $[-3,0]$ .
  - (b) (5 points) Find the number  $c$  guaranteed by the Mean Value Theorem.
2. (5 points) Water began leaking from a dam starting at noon on day 1. By noon on day 2, 24 hours later, the leak had been fixed, but the water level had fallen 48 inches. Explain how you can be sure that at some time during those 24 hours, the water level was falling at exactly 2 inches per hour.

## Final Exam Question for Round 2

(25 points) Santa is driving a snowmobile one early snowy morning on a highway with a speed limit of 65mph. A surveillance camera records his snowmobile entering a tunnel that is 0.2 miles long moving at 60mph. Another camera records the same snowmobile exiting the tunnel 10 seconds later at 60mph. (10 seconds is approximately 0.00278 hours.) Is there any mathematical evidence that Santa was speeding while in the tunnel? Justify your answer.

*Note: If you use a function  $f(t)$  in your solution, provide a verbal description of it. Be sure to reference by name any theorems you have used in your solution.*

## Rolle's Theorem and Mean Value Theorem Post-Lesson Student Survey

We asked the students for feedback about our lesson. Answers to these questions should be collected in class immediately. Allowing the students to take a survey home to bring back at a later date proved disastrous. Hardly anyone brought it back.

1. What part of the lesson was most engaging? Why?
2. What part of the lesson was most confusing? Why?
3. Please add any additional comments you have regarding the lesson.

## Rolle's Theorem/Mean Value Theorem Lesson Study Observation Form

We asked several other instructors to observe our lesson and give feedback. These are the questions we asked them to answer.

### Evidence collected:

Worksheets  
Homework  
Quizzes  
Videotape  
Observations

### Focal Questions:

*During lecture/discussion time:*

Do students seem engaged? (Participating, note taking, asking questions, visual facial cues (nodding, etc))

How many students are engaged? Who is engaged?

Who is actively participating?

*During exploration time (graphs):*

Do students understand what they are trying to draw?

Where are they getting stuck?

How did they get unstuck? (Did an instructor help them, another student, etc?)

Did the students feel they got the right picture at the end?

### **RT Graph Exploration – Day 1**

Do students understand what they are trying to draw?

Where are they getting stuck?

How did they get unstuck? (Did an instructor help them, another student, etc?)

Did the students feel they got the right picture at the end?

### **During class discussion of the graphs:**

Do students seem engaged? (Participating, note taking, asking questions, visual facial cues (nodding, etc))

How many students are engaged? Who is engaged?

Who is actively participating?

General Comments:

### **Rolle's Theorem Discussion – Day 1**

Do students seem engaged? (Participating, note taking, asking questions, visual facial cues (nodding, etc))

How many students are engaged? Who is engaged?

Who is actively participating?

General Comments:

### **MVT Graph Exploration – Day 1**

Do students understand what they are trying to draw?

Where are they getting stuck?

How did they get unstuck? (Did an instructor help them, another student, etc?)

Did the students feel they got the right picture at the end?

### **During class discussion of the graphs:**

Do students seem engaged? (Participating, note taking, asking questions, visual facial cues (nodding, etc))

How many students are engaged? Who is engaged?

Who is actively participating?

General Comments:

**RT/MVT Review and Proof of MVT – Day 2**

Do students seem engaged? (Participating, note taking, asking questions, visual facial cues (nodding, etc))

How many students are engaged? Who is engaged?

Who is actively participating?

General Comments:

**In-Class Examples – Day 2**

Do students seem engaged? (Participating, note taking, asking questions, visual facial cues (nodding, etc))

How many students are engaged? Who is engaged?

Who is actively participating?

General Comments: