Lesson Study Project in Mathematics, Spring 2010
University of Wisconsin Marshfield
University of Wisconsin Marathon County
Final Report

Date: March 30, 2010

Students: MAT 222 Calculus II students at University of Wisconsin- Marshfield
Team Members:

Kavita Bhatia-Associate Professor, UW-Marshfield

Kirthi Premadasa-Assistant Professor, UW-Marathon County

Research Theme (Broad Objectives)

- To enhance the discovery skills of students
- To enhance the pattern recognition skills of students.

1. **The unit:** Calculus II

2. **The Lesson :** Arc length of a curve as an Application of integration

3. About the Unit:

Among the topics covered in MAT 222, Calculus II are the different applications of integration, such as volume, work, average value, arc length, pressure and surface of revolution. The typical approach to each of these applications is to introduce the relevant Riemann sum and the derivation of the particular formula for the concept involving the integral. However, some students simply remember the final formula but do not obtain insight as to why the formula works or how it was derived. This is a valuable insight lost in a higher-level course such as Calculus II. Students encounter a number of Riemann Sum setups such as area, volume, average value and work. However, these Riemann Sums are introduced independently to represent the different scenarios and a "capstone" lesson which brings them all together is absent from the typical Calculus II syllabus though professors do make the attempt to show the underlying common theme to the students, every time a Riemann sum is introduced. Students have also by this time, learnt most of the techniques used in integration that are covered in Calculus II, so once the integral is setup for the practical application, they will have the tools to evaluate it.

4. Goals of the lesson

a) Students will learn how to make a manual calculation of a Riemann sum for the arc length of a given sample curve using a few subdivisions.

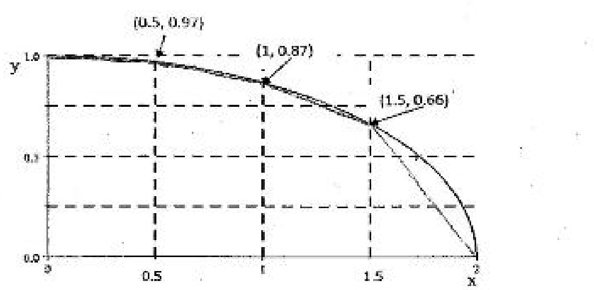
b) Students will use the knowledge obtained through the Riemann Sum -> Integrations models that they have seen before, to "discover" an integration formula for the arc length of any continuous curve.

c) Students will use the formula that they "discovered", together with the integration techniques, taught in the course to evaluate the actual arc length of the curve.

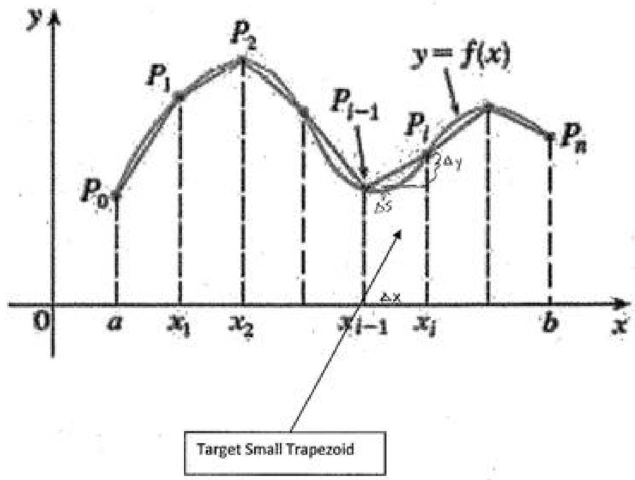
d) Students will understand the underlying theme behind all the Riemann sums that they have encountered.

5. Lesson Plan for research Lesson 1

Delivered by : Kavita Bhatia
Observers: Kirthi Premadasa
Date/Time: March 16,2010 10.00 a.m.-10.50 a.m.
Location: University of Wisconsin -Marshfield

Time	**Activity**	**Anticipated Student Response**
First 5 minutes	Instructor does a quick regress of Riemann sums and outlines objectives of Lesson. Breaks the class in to 4 groups and hands out task 1	
Next 15 minutes	Students work on task 1 which is shown below. I) We shall start by estimating the length of the arc of the ellipse $y = \sqrt{1-x^2/4}$, $0 \le x \le 2$ by using 4 subdivisions. Given below is the graph of the function along with the partition points. Working with your group members, estimate the length of this curve by the lengths of the line segments connecting the partition points.  Instructor will request students who finish the task to write the answer on the board.	We expect the students to spend some time trying to find the technique needed to find the distance between two points. We expect them to either to use the distance formula or the Pythagorean principle. Except , for that we except the students to do the task smoothly
Next 5 minutes	Instructor presents the calculation for 8 subdivisions to the students to reinforce their calculation for 4 subdivisions. The instructors goes on to	Students see how the value of the

	explain that in task 2, the students will get a chance to actually "discover" the integral formula for the arc length using the knowledge gained from previous Riemann sums they have seen , together with the guidance of the worksheet. Instructors hands out the worksheet for task 2	arc length limits when the number of subdivisions is increased
Nest 15 minutes	Students work on the worksheet for task 2 which is shown below.	We expect the students to do the first three "labeling tasks" smoothly without much difficulty

	Task: Finding an Integral formula for the arc length of a curve.  Our task is to get a formula for the arc length of the given graph using integration . As we did before, **we will use the following strategy** Arc Length = Sum of Small Arc Lengths $= \sum$ (small arc lengths) $= \sum(\ \ )\Delta x = \int (\ )dx$ *Step 1:* Let Δx denote the width of the "Target small trapezoid". Label Δx in the diagram. *Step 2:* Let Δy denote the difference in the heights of the "Target small trapezoid". Label Δy in the diagram. *Step 3:* Let Δs denote the small arc length (in the vicinity of the Target small trapezoid) . Label Δs in the diagram.	
	Step 4: Since Δx is small, write down the name of a line segment whose length will approximate Δs $\Delta s \approx$. .... *Step 5:* Using Δx and Δy, find an approximation for the length of the line segment you found above $\Delta s \approx$:...........,.......................... *Step 6:* Now substitute this expression for Δs in the formula Big Arc length = $\sum \Delta s$) Big Arc length = $\sum$...........	Using their prior experience with Riemann sums, we expect students to see that Δs can be approximed by the hypotenuse of the traingle with Δx and Δy as sides. We expect step 6 to be the crucial step and expect students to get $\Delta s^2 = \Delta x^2 + \Delta y^2$

	Step 7: Now "factor out the Δx " from your expression and write this summation in the form Big Arc length = $\sum \Delta s = \sum(\quad)\Delta x$ Big Arc length = $\sum(\quad\quad\quad)\Delta x$ *Step 8:* When Δx is very small, this is nothing but an integral. Write what you stated above as an integral. Change the Δx to dx and change Δy to dy. Big Arc length = $\int_a^b (\quad\quad\quad)dx$ *Step 9:* Check whether your formula is correct by looking at Page 526 of your Book (Chapter 8.1 , formula 3)	Once the students get step 6, we expect them to factor out Δx and get the answer $\Delta s = \Delta x \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$ And the corresponding formula for the arc length. However since the final evaluation in step 9 involves trigonometric substitutions, we expect the students to have some difficulty during the evaluation

See Appendix 2 for a copy of the complete lesson.

6. Minutes of Research Lesson 1

Lesson Study on Integration Applications

Kavita Bhatia (UW-Marshfield)

Kirthi Premadasa (UW-Marathon County)

Research lesson

Date: 3/16/2010

Room 321. UW-Marshfield/Wood County

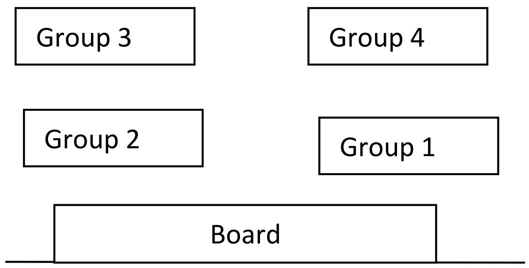
Instructor: Kavita Bhatia

1) **Lesson Progression**

Kavita started the lesson by saying that we are planning to do a lesson study on the "Arc Length" length lesson on Lesson Study. She said that one of the objectives of the Lesson Study would be to "discover" the Arc Length formula. She then went on to say that just as we did area and Volume, we will be setting up Riemann sums.

Then Task 1 was handed out and students were asked to break in to groups. The students broke in to 4 groups. The groups were names Group 1, Group 2, Group 3 and group 4.

The following diagram represents how the groups were located.



Task One progress

Start at 10.05	Group 1	Group 2	Group 3	Group 4
By 10.07 a.m.	0	0	0	5% (Talking about distance formula)
(By 10.10 a.m.) (Kavita gives reiterates the instruction in the worksheet about "finding the length of line segment") First 5 minutes	30% (Student use Pythagorean rule to find distance)	30% (Student use Pythagorean rule to find distance)	60% (Student use Pythagorean rule to find distance)	10% (Student use Pythagorean rule to find distance)
(By 10.15 a.m.)	60%	60%	100%	40%
(By 10.20 a.m.)	100%	100%	100%	80%
(By 10.22 a.m.)	100%	100%	100%	100%

At this point, Kavita said that to the class that they are now going to "discover "an integral formula for the arc length using a Riemann sum argument similar to what was done for areas and volumes.

She hands out task 2.

Task 2 progress

	Group 1	Group 2	Group 3	Group 4
By 10.25 a.m.	30%	30%	30%%	30%
By 10.30 a.m.	60%	60%	60%	60%
By 10.35 a.m. All groups are	60%	60%	60%	70% Had something

stuck on Step 7 Next five minutes (Did not seem to understand how to "factor out a variable" when it is not a "common factor").				like $\sqrt{\Delta x}(\ldots)$
By 10.40 a.m. To break the deadlock in Step 7, Kavita asks the students "How do you factor a 5 out of $5 + x$?" This triggers all groups to correctly factor out the Δx from $\sqrt{(\Delta x)^2 + (\Delta y)^2}$	85% (first to finish Step 8)	85% (next to finish step 8)	85%	80%
By 10.45	Evaluating the integral 95%	Evaluating the integral 95%	90% (Made a mistake in the integral by thinking $\left(\frac{dy}{dx}\right)^2$ as $\frac{d^2y}{dx^2}$	90%
By 10.50	100%	100%	100%	100%

2) Observations for Task 1

- One student in Group 3 finished the full task within 10 minutes. Others followed suit.
- Group 4 was the first to have the idea of using the distance formula to find the length but they could not carry it through
- Student from Group 1 presented the calculation on the board.
- All groups ended up getting the same answer.

3) Observations for Task 2

- All groups readily progressed up to step 6, showing that they were quite familiar with terms like Δx, and Δy which appeared in the Riemann Sum applications that they have met before.

- The main reason for all groups getting stuck in step 7 is interesting. It is most likely that students perceive the term "factor" only when there is a common fact and not very fluent in the algebraic process of factoring out a term which is not a common term.
- All groups made an impressive performance by "discovering" the arc length formula.
- The engagement level of all groups was high.
- Group 3's performance was somewhat dominated by one student. The good side of this was they managed to finish task one the fastest. However, when that student made an error in the last step of task 2, the entire group had a setback.

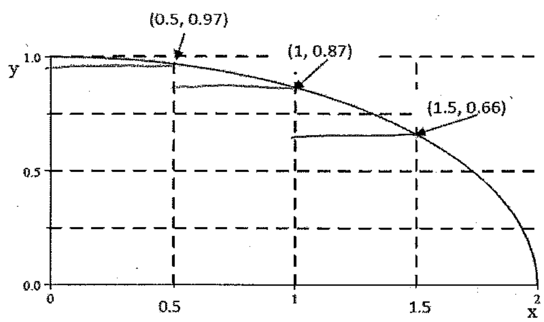
7. Appendix 1: Sample Student work

Group 1

Arc Length

Our goal in this section is to find an integral formula to calculate the arc length of a curve described by the equation $y = f(x)$, $a \leq x \leq b$ by using the definition of the integral as a limit of Riemann sums.

I) We shall start by estimating the length of the arc of the ellipse $y = \sqrt{1 - x^2/4}$, $0 \leq x \leq 2$ by using 4 subdivisions. Given below is the graph of the function along with the partition points. Working with your group members, estimate the length of this curve by the lengths of the line segments connecting the partition points.



.03, .5, X = .5008991915

.1, .5, x = .5099019514

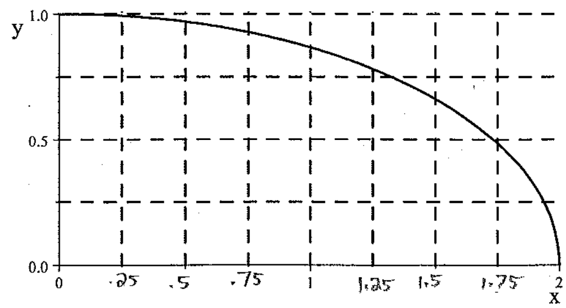
.21, .5, x = .5423098745

.66, .5, x = .8280096618

2.381120679

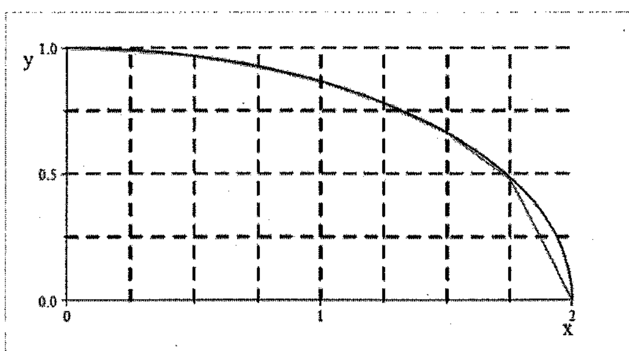
Group 1

II) Below is graph of the arc of the ellipse $y = \sqrt{1 - x^2/4}$, $0 \le x \le 2$, on 8 subdivisions.



$$y = \sqrt{1 - \frac{x^2}{4}}$$

x	0	.25	.5	.75	1	1.25	1.5	1.75	2
y	1	.99216	.96825	.92702	.86603	.78062	.66144	.48412	0



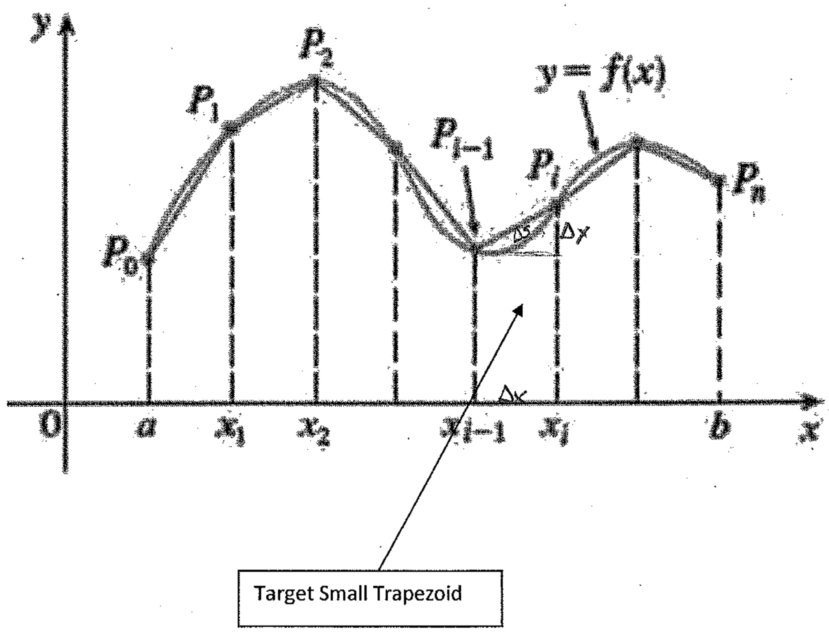
On each subinterval I have estimated the arc by a line segment. The length of the line segment is an estimate for the arc length on the corresponding subinterval. The sum of the lengths of the line segments is an estimate for the length of the arc.

Sub – interval	1	2	3	4	5	6	7	8
Length	.2502	.2508	.2534	.2573	.2642	.2770	.3065	.5449

Sum of the lengths of the line segments = 2.4043

Group 1

Task: Finding an Integral formula for the arc length of a curve.



Our task is to get a formula for the arc length of the given graph using integration . As we did before, **we will use the following strategy**

Arc Length = Sum of Small Arc Lengths

$= \sum$ (small arc lengths) $= \sum(\quad)\Delta x = \int (\quad) dx$

Step 1: Let Δx denote the width of the "Target small trapezoid". Label Δx in the diagram.

Step 2: Let Δy denote the difference in the heights of the "Target small trapezoid". Label Δy in the diagram.

Step 3: Let Δs denote the small arc length (in the vicinity of the Target small trapezoid) . Label Δs in the diagram.

Step 4: Since Δx is small, write down the name of a line segment whose length will approximate Δs

$\Delta s \approx$ ... $P_{i-1} - P_i$ ...

Step 5: Using Δx and Δy, find an approximation for the length of the line segment you found above

$\Delta s \approx$ ... $\sqrt{\Delta x^2 + \Delta y^2}$ ...

Step 6: Now substitute this expression for Δs in the formula

Big Arc length = $\sum \Delta s$

Big Arc length = $\sum$ ... $\sqrt{\Delta x^2 + \Delta y^2}$ ... $\sqrt{\Delta x^2\left(1 + \frac{\Delta y^2}{\Delta x^2}\right)}$ $\sqrt{\Delta x^2}\sqrt{1 + \frac{\Delta y^2}{\Delta x^2}}$

Step 7: Now "factor out the Δx " from your expression and write this summation in the form

Big Arc length = $\sum \Delta s = \sum(\quad)\Delta x$

Big Arc length = $\sum\left(\sqrt{\left(1 + \frac{\Delta y^2}{\Delta x^2}\right.}\right)\Delta x$

Step 8: When Δx is very small, this is nothing but an integral. Write what you stated above as an integral. Change the Δx to dx and change Δy to dy.

Big Arc length = $\int_a^b \left(\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}\right) dx$

Step 9: Check whether your formula is correct by looking at Page 526 of your Book (Chapter 8.1 , formula 3)

Group 3

Arc Length

Our goal in this section is to find an integral formula to calculate the arc length of a curve described by the equation $y = f(x)$, $a \leq x \leq b$ by using the definition of the integral as a limit of Riemann sums.

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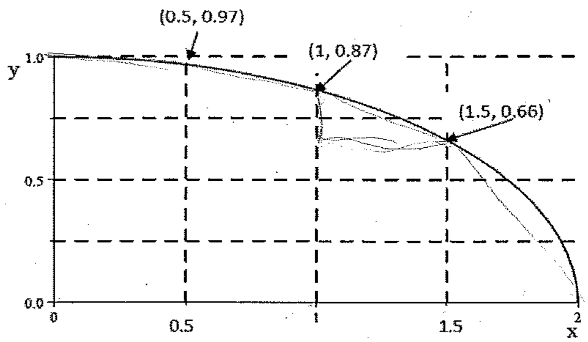


$$\int_0^2 \sqrt{1 + \left(-\frac{x}{4}\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}\right)^2}$$

↓

$$\int_0^2 \sqrt{1 + \frac{x^2}{16}\left(1-\frac{x^2}{4}\right)^{-1}}$$

$\sqrt{\Delta x^2 + \Delta y^2}$

↓

	Δx	Δy	l/b
1	.5	-.03	.5008992
2	.5	-.1	.5099195
3	.5	-.21	.542399
4	.5	-.66	.8280066

2.3812

$$\int_0^2 \sqrt{1 + (f'(x))^2}\, dx$$

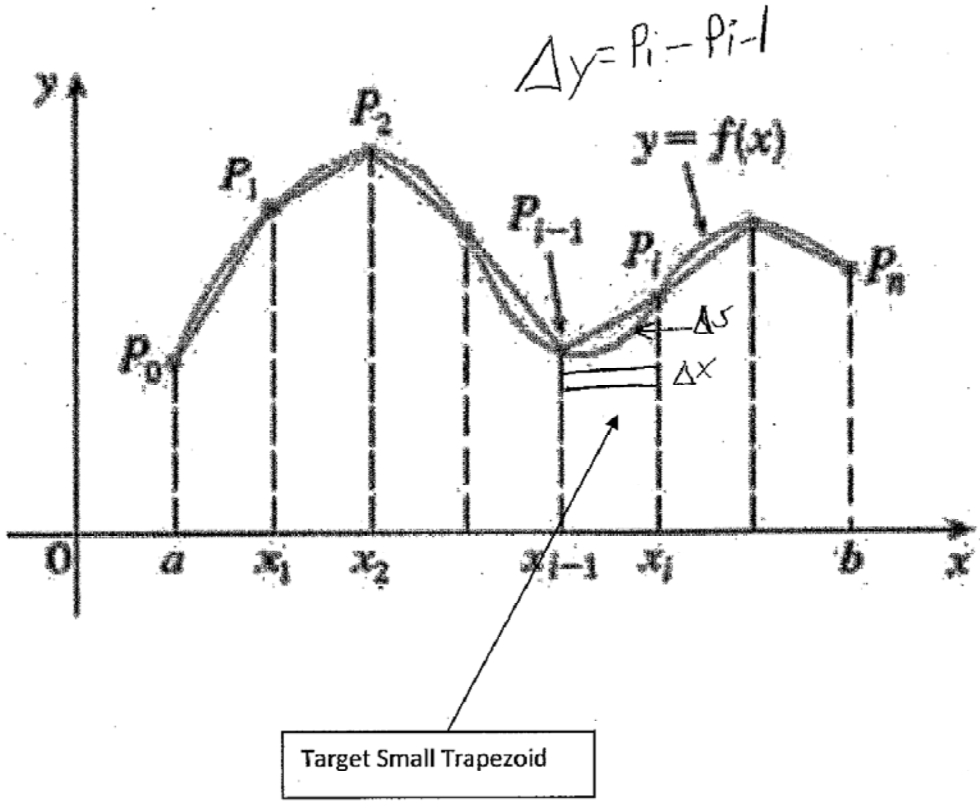
$y = (1 - x^2/4)^{\frac{1}{2}}$

$y' = \frac{1}{2}\left(1 - \frac{x^2}{4}\right)^{-\frac{1}{2}}(-x/2) = -\frac{x}{4}\left(1 - \frac{x^2}{4}\right)^{-\frac{1}{2}}$

$y'' =$

Gr 3

Task: Finding an Integral formula for the arc length of a curve.



Our task is to get a formula for the arc length of the given graph using integration . As we did before, **we will use the following strategy**

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$\Delta s \approx$ ...... $P_i - (P_{i-1})$ ....

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$\Delta s \approx$ ...... $\sqrt{\Delta x^2 + \Delta y^2}$ ....

Step 6: Now substitute this expression for Δs in the formula

Big Arc length = $\sum \Delta s$

Big Arc length = $\sum$ .. $\sqrt{\Delta x^2+\Delta y^2}$ .... $\sqrt{1+\frac{\Delta y^2}{\Delta x^2}}\ \Delta x$ $\quad \Delta s = \sqrt{\Delta x + \Delta y^2}$ $\quad \left(\frac{dy}{dx}\right)^2$

Step 7: Now "factor out the Δx " from your expression and write this summation in the form

Big Arc length = $\sum \Delta s = \sum(\quad)\Delta x$

Big Arc length = $\sum_{i=1}^{n}(\sqrt{1+\frac{\Delta y^2}{\Delta x^2}} \quad)\Delta x$

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Big Arc length = $\int_a^b (\sqrt{1+\frac{dy^2}{dx^2}} \quad)dx$

Step 9: Check whether your formula is correct by looking at Page 526 of your Book (Chapter 8.1 , formula 3)

8. Appendix 2: Arc Length Lesson

Arc Length

Our goal in this section is to find an integral formula to calculate the arc length of a curve described by the equation $y = f(x),\ a \le x \le b$ by using the definition of the integral as a limit of Riemann sums.

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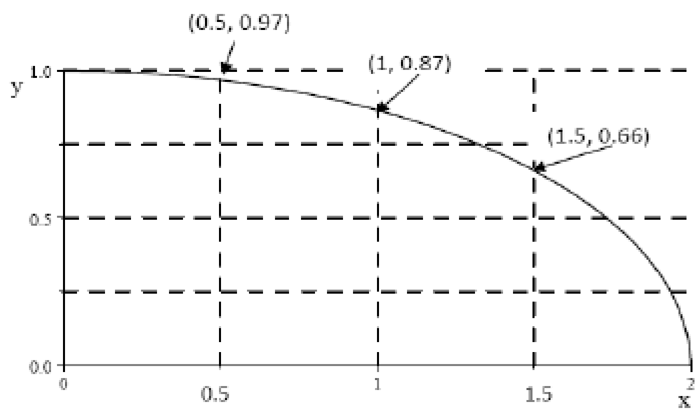

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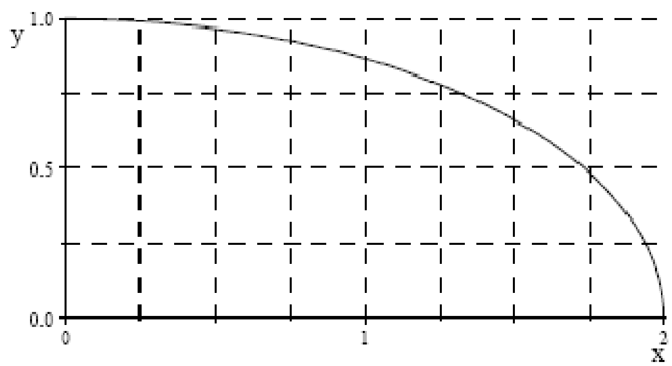
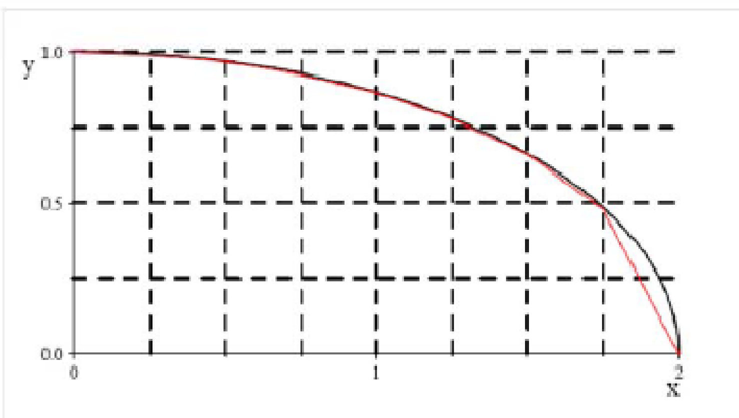


$$y = \sqrt{1 - \frac{x^2}{4}}$$

x	0	.25	.5	.75	1	1.25	1.5	1.75	2
y	**1**	.99216	.96825	.92702	.86603	.78062	.66144	.48412	0



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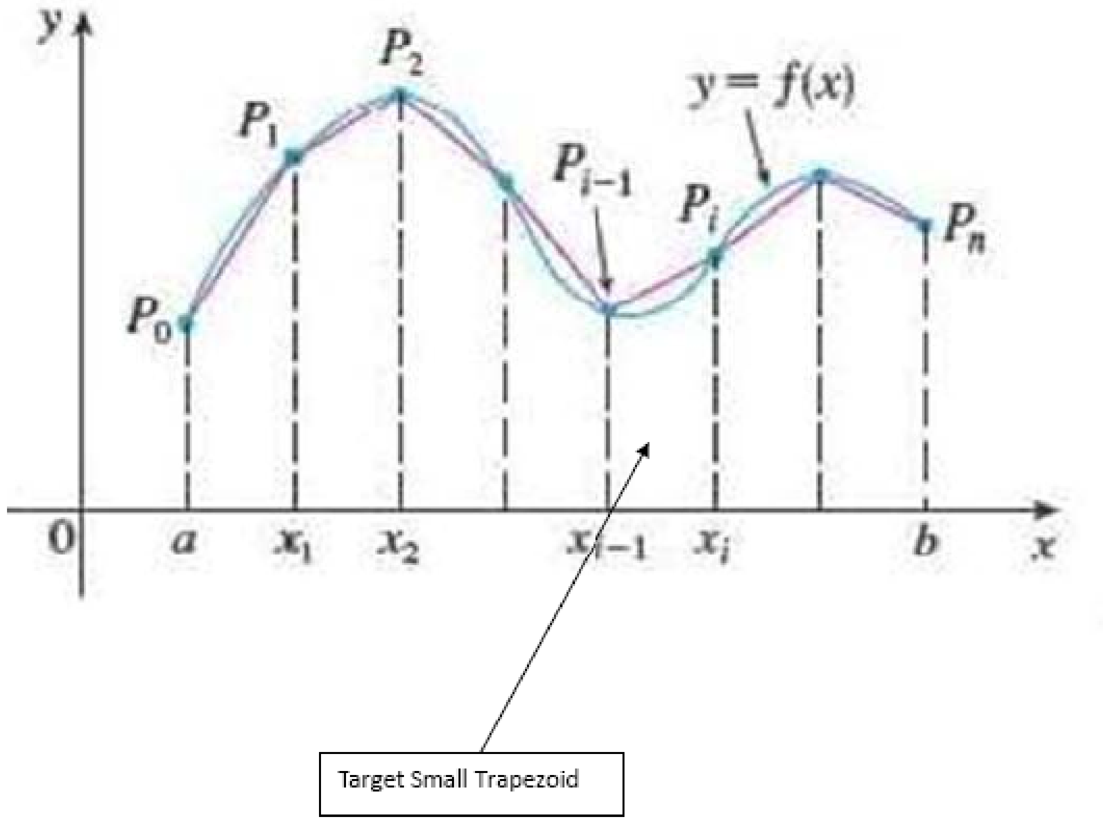

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$\Delta s \approx$......................................

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Big Arc length = $\int (\qquad\qquad)dx$

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